Momentum Integral Methods for the Laminar Free Shear Layer

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A momentum integral technique is used to describe the nonsimilar growth of the constant pressure laminar free shear layer with finite initial thickness An essential feature of the analysis is the division of the shear layer into two parts, one above and one below the zero streamline Separate polynomial or exponential profiles are used to represent the velocity profile in each part; suitable matching conditions lead to closed form solutions of the momentum equation with simple expressions for velocity profiles Use of the Howarth transformation allows a description of the effects of Mach number and boundary temperature on the growth of the shear layer and the Mach number M_* along the dividing streamline

Formulation of the Problem

MOMENTUM integral methods are very convenient for treating a wide variety of separated flow problems Several examples have been reported in the literature[1-5] and further applications are being pursued [6,7] For the constant pressure mixing problem, the application is simple and direct, and leads to closed form solutions of the momentum equation with simple profiles This paper describes the development of a constant pressure laminar free mixing layer with a finite initial thickness

We assume that the usual boundary-layer approximations are valid in the mixing layer, and the Howarth transformation is used to reduce the equations to incompressible form (unbarred variables are in the transformed plane):

$$s = \bar{s} \qquad y = \int_0^{\bar{y}} \left(\frac{\rho}{\rho_e}\right) d\bar{y} \tag{1}$$

where $\bar{s}$ is the distance measured along the dividing streamline, and $\bar{y}$ is the distance normal to the dividing streamline Defining the stream function ψ in the usual manner,

$$\rho\bar{u} = \rho\,(\partial\psi/\partial\bar{y}) \qquad \rho\bar{v} = -\rho\,(\partial\psi/\partial\bar{s})$$

we have

$$u = (\partial\psi/\partial y) \qquad v = -(\partial\psi/\partial s) \tag{2}$$

which serves to define the transformed velocities u and v ψ automatically satisfies the continuity equation, and the zero streamline $\psi = 0$ coincides with the x axis ($y = 0$)

To reduce the problem to its simplest elements, the approximations are made so that the density-viscosity product $\rho\mu$ is constant and the Prandtl number is unity The continuity, momentum, and energy equations in the transformed variables (s, y, u, v) are now identical to their incompressible form:

$$(\partial u/\partial s) + (\partial v/\partial y) = 0 \tag{3}$$

$$u(\partial u/\partial s) + v(\partial u/\partial y) = \nu\,(\partial^2 u/\partial y^2) \tag{4}$$

$$u(\partial H/\partial s) + v(\partial H/\partial y) = \nu_e(\partial^2 H/\partial y^2) \tag{5}$$

where ν_e is the kinematic viscosity (μ/ρ) Equations (4) and (5) indicate that $H = A + Bu$ is a solution of the energy equation (5), where A and B are constants This solution is admissible only if the initial and boundary conditions permit! The momentum equation (4) is now independent of the energy equation (5)

Two-Layer Method

One method of solution is to integrate the equations numerically, starting with an assumed velocity profile at $s = 0$ The calculation is continued until a similar profile corresponding to $s \to \infty$ is reached This technique was used by Denison and Baum [8] The approximate solution adopted here is to represent the velocity profile by a simple analytic function containing several parameters that are allowed to vary with s By multiplying the s momentum equation by $u^j (j = 0, 1, 2 \ldots)$ and integrating across the shear layer, coupled ordinary differential equations are obtained which describe the variation of the velocity profile parameters in the s direction Boundary conditions are also applied at the extremities of the shear layer The total number of boundary conditions and moment equations must be equal to the number of parameters appearing in the velocity profile

In the present problem, however, the initial velocity profile is so radically different from subsequent profiles that it is difficult to represent the velocity profile as a whole by a single expression Hence, the shear layer is divided into two layers: region I, $y > 0$; and region II, $y < 0$ The velocity profile in each layer is assumed to be of the form:

Region I

$$\frac{u}{u} = f = A_0 + e^{-\sigma\eta} \sum_{k=0}^{m} a_k \eta^k \qquad \eta = \frac{y}{\delta_1} \tag{6}$$

Region II

$$\frac{u}{u} = g = B_0 + e^{\sigma\mathfrak{y}} \sum_{k=0}^{n} b_k \mathfrak{y}^k \qquad \mathfrak{y} = \frac{y}{\delta_2} \tag{7}$$

$$u(s) = u_*(s) \text{ along } \psi = y = 0$$

and $\sigma = 0$ or 1 depending upon the type of representation that is desired The number of terms (m, n) are chosen so that the resulting profile will provide a "good approximation"

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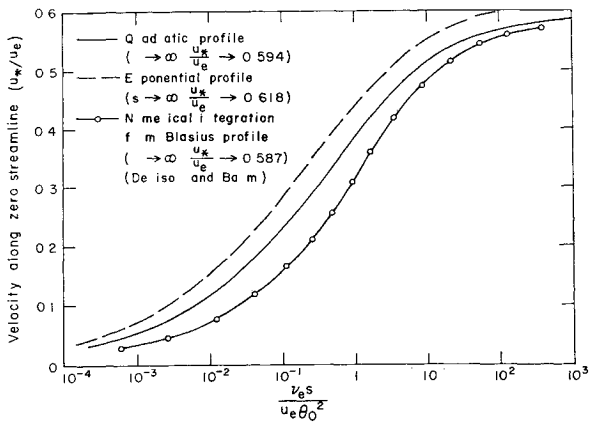

Fig 1 Effect of shear layer profile on velocity along zero streamline

for the physical problem being considered ‡ The profile parameters (A_0, a_k, B_0, b_k, δ_1, δ_2) are determined by the boundary conditions at the upper and lower extremities of the shear layer, the matching conditions at $y = 0$, and the moment equations The *possible* boundary conditions are:

Outer Boundaiy

$$\begin{array}{ll} \sigma = 0 & \eta = 1 \\ \sigma = 1 & \eta \to \infty \end{array} \qquad = 1, f' = f'' = \quad = 0 \qquad (8)$$

Inner Boundary

$$\begin{array}{ll} \sigma = 0 & \mathfrak{y} = -1 \\ \sigma = 1 & \mathfrak{y} \to -\infty \end{array} \qquad g = 0, g' = g'' \quad = 0 \qquad (9)$$

(Primes denote derivatives of f with respect to η and derivatives of g with respect to $\mathfrak{y}$) By demanding continuity of the velocity and its derivatives at $y = 0$, the following matching conditions are obtained:

$$(\partial f/\partial \eta)_0 = (\delta_1/\delta_2)(\delta g/\delta \mathfrak{y})_0 \qquad r = 0, 1, 2, \qquad (10)$$

Solution for a Quadratic Profile

A very simple example of this method is a quadratic pio-file Let $\sigma = 0, m = n = 2$, so that the eight unknowns aie§ ($a_0, a_1, a_2, b_0, b_1, b_2, \delta_1, \delta_2$) The momentum integral equations are

$$\frac{d}{ds}\left[\delta_1 \int_0^1 f(1-f)d\eta\right] = \frac{\nu_e f'(0)}{\delta_1 u_e}$$

$$\frac{d}{ds}\left[\delta_2 \int_{-1}^0 g^2 d\mathfrak{y}\right] = \frac{\nu_e g'(0)}{\delta_2 u_e} \qquad (11)$$

Define the initial thickness δ_0 and the quantities λ and δ by $\lambda = 1 - (u^*/u_e)$ and $\delta = \delta_1 + \delta_2$, where the limit $s \to 0$ corresponds to $\lambda \to 1$ and $\delta \to \delta_1(s = 0) \equiv \delta_0$ Then the quadratic profiles satisfying the boundary and matching conditions $r = (0, 1)$ are

$$f = 1 - \lambda(1-\eta)^2 \qquad g = (1-\lambda)(1+\mathfrak{y})^2 \qquad (12)$$

‡ One difficulty with the momentum integral method is that the number of terms taken to represent the velocity profile is arbitrary In this case, there are two criteria that aid in this choice: 1) the assumed profile must correspond closely to the specified initial profile as $s \to 0$; and 2) the final similar profile for $s \to \infty$ must agree well with the exact solution ob tained by Chapman [9] Because of its simplicity, the free shear-layer problem represents an excellent test case for the moment integral method, since various combinations of profile parameters, boundary conditions, and moment equations may be used without undue algebraic complication Such an exhaustive examination, however, is not attempted in this investigation

§ The quantities a_0 and b_0 are redefined to include A_0 and B_0

Table 1 Shear rates expressed in normalized terms

	Blasius	Quadratic	Exponential
$(\theta_0/u)(\partial u/\partial y)_0$	0 2205	0 2267	0 5

and the thicknesses are related by $\delta_1 = \lambda\delta$ and $\delta_2 = (1 - \lambda)\delta$ Substituting Eq (12) into Eq (11) and solving the iesulting equations, we obtain

$$\frac{\nu e^s}{u\,\delta_0^2} = \frac{1}{5}\frac{(1-\lambda)^3}{(3 - 9\lambda + 4\lambda^2)^2} - \frac{1}{55}\frac{(\lambda+3)(1-\lambda)}{(3-9\lambda+4\lambda^2)} + \frac{4}{55\,(33)^{1/2}} \times \left[\log\left|\frac{8\lambda - 9 + (33)^{1/2}}{8\lambda - 9 - (33)^{1/2}}\right| + \log\left(\frac{(33)^{1/2}+1}{(33)^{1/2}-1}\right)\right] \qquad (13)$$

which has the asymptotic form $\lambda = 0\,406$ as $s \to \infty$ This iesult, $(u_*/u_e) = 0\,594$, is extremely close to the exact value of 0 587 found by Chapman [9] The initial profile is, of couise, restricted to a quadratic with $\lambda = 1$ A more accurate solution would be one in which two or more free parameters aie available to specify the initial profile; the important effect to the initial pi ofile on the rate of growth of u_* would then be apparent immediately

Solution for an Exponential Profile

The exponential profiles ($\sigma = 1$) provide a particularly simple system of matching relations Because of the fact that all derivatives of u approach zero at $\pm\infty$, the unknown parameters ($\delta_1, \delta_2, a_k, b_k$) are determined only by the matching conditions at $y = 0$ and the moment equations $M_j(\pm)$

The velocity profiles for $\sigma = 1$ and $m = n = 0$, satisfying the matching conditions $r = 0, 1$, are $y > 0, f = 1 - \lambda e^{-\eta}$; $y < 0$, $g = (1 - \lambda)e^{\mathfrak{y}}$; and the shear layer thicknesses are ielated by $\lambda\delta_2 = (1 - \lambda)\delta_1$ Substituting this velocity pi o-file into Eq (13) [using appropriate limits of integration], the solution is found to be

$$\frac{\nu_e s}{u_e \delta_0^2} = \frac{3}{20} + \frac{(4\lambda - 1)}{20X} + \frac{\lambda(1-\lambda)}{4X^2} + \frac{3}{10(5)^{1/2}}\left[\log\left|\frac{2\lambda - 3 + 5^{1/2}}{2\lambda - 3 - 5^{1/2}}\right| + \log\left|\frac{1+5^{1/2}}{1-5^{1/2}}\right|\right] \qquad (14)$$

$$X = 1 - 3\lambda + \lambda^2$$

whereas the similarity form for the shear layer thickness is

$$\lim_{s \to \infty} (\delta/x)(Re\ )^{1/2} = 4\,10$$

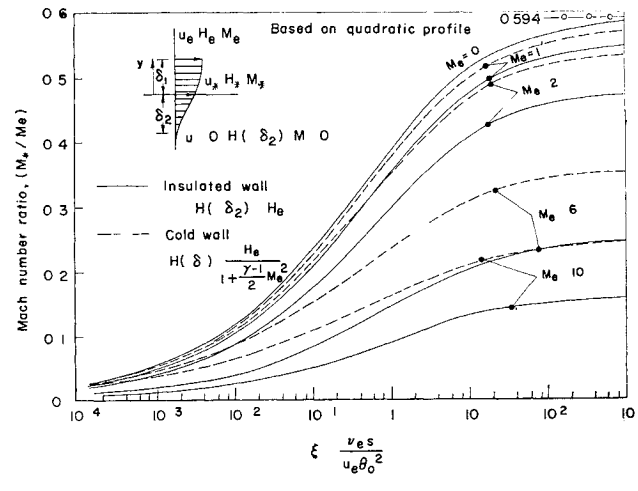

Fig 2 Effect of Mach number and internal cooling on M_*

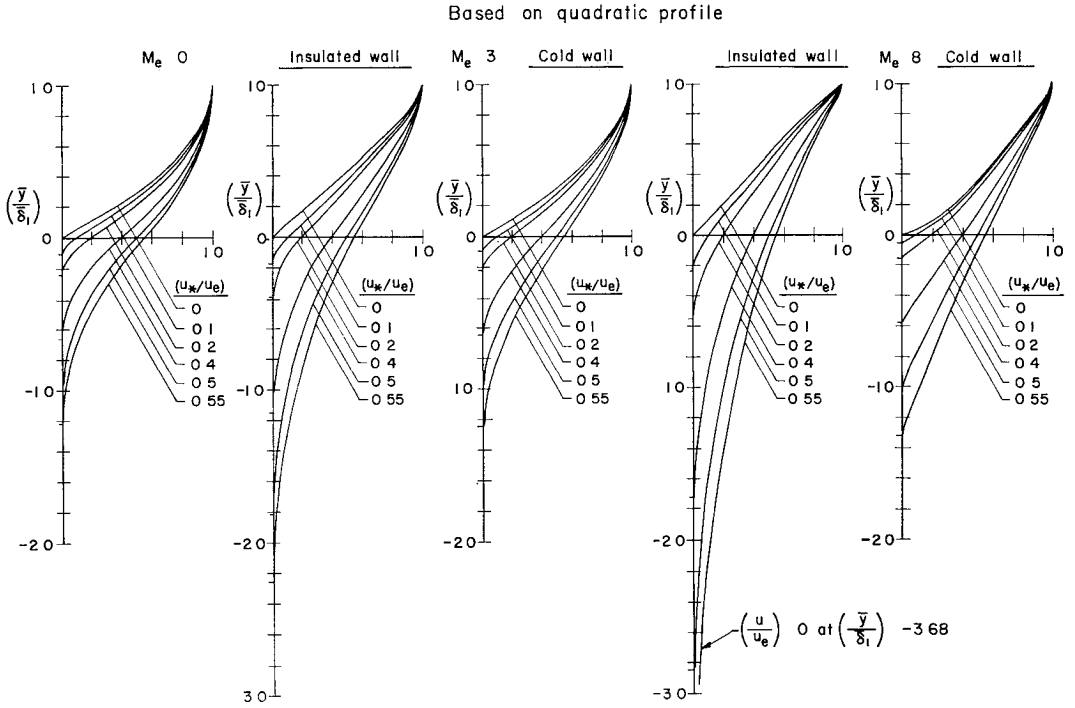

Fig 3 Velocity profiles in the compressible shear layer

and the asymptotic velocity ratio (u_*/u_e) is 0 618 This solution also contains a single profile parameter λ, so that the initial profile for $s \to 0$ and $\eta > 0$ is constrained to be $(u/u_e) = 1 - \exp(-\eta)$

Transformation to the Compressible Plane and Comparison with Numerical Solutions

The initial momentum thicknesses in the incompressible and physical coordinates are identical in form, namely,

$$\bar{\theta}_0 = \int_0^\infty \left(\frac{\rho\bar{u}}{\rho\,\bar{u}}\right)\left(1 - \frac{\bar{u}}{\bar{u}_e}\right) d\bar{y} = \int_0^\infty \left(\frac{u}{u_e}\right)\left(1 - \frac{u}{u_e}\right) dy = \theta_0 \tag{15}$$

It is for this reason that the momentum thickness appears to be the "proper" length scale of the initial profile The unbarred form of θ_0 will be used in a subsequent discussion

The quadratic profile is found to give $\theta_0 = \frac{2}{15}\delta_0$, and for the simple exponential profile $\theta_0 = \frac{1}{2}\delta_0$ For the numerical solution of Denison and Baum,[8] which uses a Blasius initial profile, the streamwise coordinate s^* is related to the present solutions by the equation[10]

$$s^* = 0\,4863(\nu\, s/u_e\theta_0^2) \tag{16}$$

The variation of the mixing velocity u_* with the "proper" distance variable $\xi = (\nu\, s/u\,\theta_0^2)$ is shown in Fig 1 The quadratic, exponential, and Blasius results are seen to be in qualitative agreement, although the magnitude of u_* differs

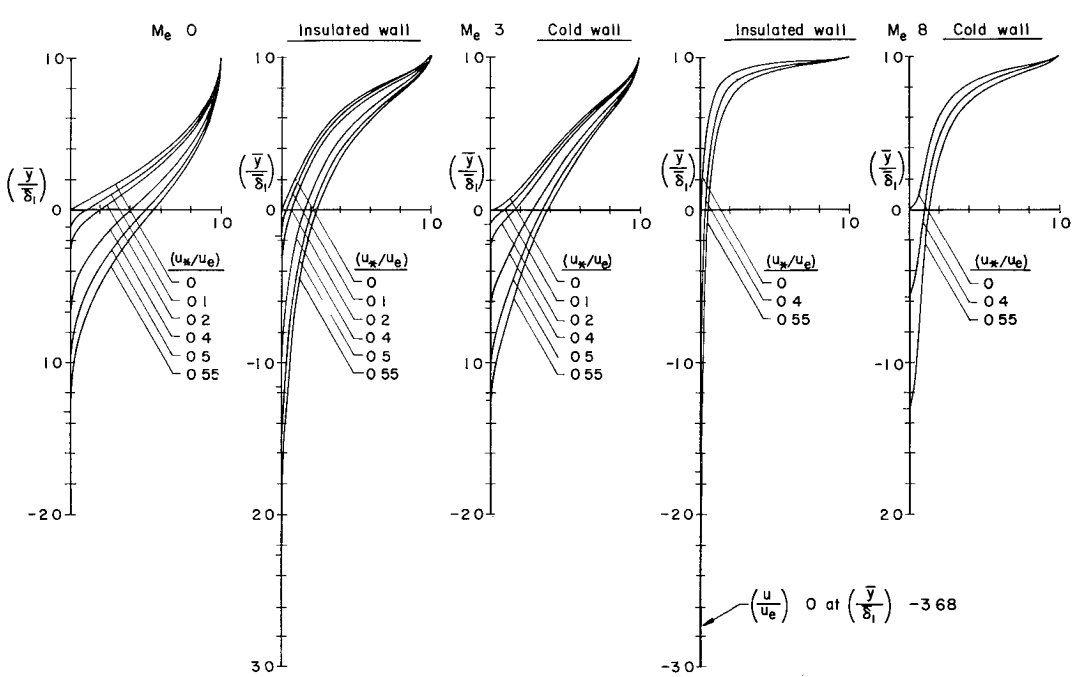

Fig 4 Mass-flow profiles in the compressible shear layer

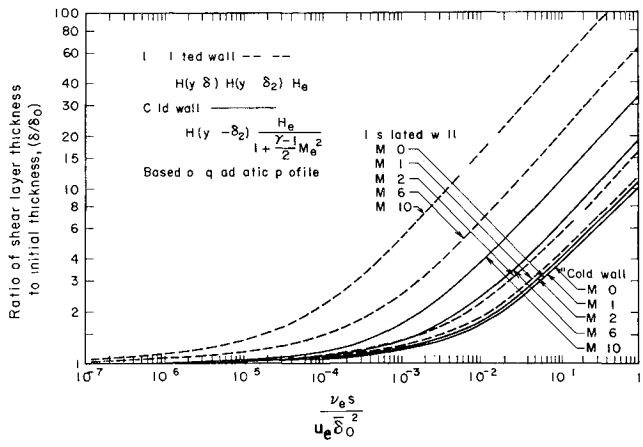

Fig 5 Effect of external Mach number and internal cooling on physical shear layer thickness

by as much as 0 15 at a given value of ξ Even more disturbing is the fact that the values of ξ for a given u_* may differ by as much as a factor of 6 Since each of the solutions involves only one free profile parameter, it may not be stated with certainty whether the differences between the numerical solution and the moment method results are produced by the differences in the initial profile or the differences in the method of calculation

However, near $\xi = 0$, the situation is quite clear Each initial profile is linear near $y = 0$, so that the initial shear $(\partial u/\partial y)_{y=0}$ determines the variation of $u_*(\xi)$ for small s These shear rates are expressed in normalized terms as shown in Table 1

From this table, it is obvious that the initial growth of $u_*(\xi)$ will be more rapid if the initial profile is an exponential than if it is a quadratic; this supposition is borne out by Fig 1 The difference between the initial shear $(\theta_0/u_e)(\partial u/\partial y)_0$ of a Blasius profile and a quadratic is small; hence, the difference between the values of $u_*(\xi)$ and for small ξ calculated by Denison and Baum[8] and the values found from the quadratic solution are attributed to the approximate nature of the momentum integral method Considering the simplicity of the momentum integral method, the agreement is very good

Effects of Heat Transfer and Calculations of M_*

For purposes of illustration, we will consider only the solution obtained for the quadratic profile Consider first the case where the temperature in the base flow region ($u \to 0$) is equal to the total temperature of the inviscid flow (this will be called the "insulated wall" case) Then $H = H_e$ through the shear layer and

$$T/T = 1 + [(\gamma - 1)/2]M_e^2[1 - (u/u)^2] \qquad (17)$$

If the fluid in the base flow region is very cold, then M_* will be large For example, consider the base temperature to be equal to the static temperature T_e of the external flow (this will be called the "cold wall" case) Then the equation corresponding to (17) is

$$T/T_e = 1 + [(\gamma - 1)/2]M_e^2[(u/u) - (u/u)^2] \qquad (18)$$

The physical distance $\bar{y}$ may be written as a function of the transformed distance y

$$\bar{y} = \int_0^y \frac{T}{T_e}\, dy \qquad (19)$$

The Mach number $M_*(\xi)$ may be calculated from Eqs (13, 17, and 18) using the definition $M_* = u_*/(\gamma R T_*)^{1/2}$ The results of this computation are shown in Fig 2 Mass-flow and velocity profiles are shown in Figs 3 and 4 In Fig 5, the physical shear layer thickness is shown as a function of the streamwise coordinate $(\nu s/u \bar{\delta}_0^2)$ These four figures lead to the following conclusions:

1) The Mach number M_* increases with increasing M, but this increase becomes small as M_e becomes large

2) At higher Mach numbers, say $M \gtrsim 6$, the difference between the values of M_* for the cold wall and insulated wall cases may be 50% or more In applying these results to the near wake of a body at high speeds, one common assumption[11] is that the turning of the shear layer at the neck of the wake is determined by the pressure rise¶ $(p' - p)$ required to bring the fluid along the zero streamline to rest by an isentropic compression process This leads to the equation

$$\frac{p'}{p} = \left(1 + \frac{\gamma - 1}{2} M_*^2\right)^{\gamma/(\gamma-1)}$$

The strong effect of base cooling on the dynamics of the near wake is therefore evident

3) The velocity and mass-flow profiles in the shear layer exhibit the same compressible-incompressible relation as the corresponding profiles for a boundary layer As M_e increases, the velocity profile in the physical plane becomes more linear The mass-flow profile becomes less full with increasing M_e, and for the "insulated wall" temperature boundary condition, the mass flow below the zero streamline ($y < 0$) is small

4) The shear layer penetrates more and more deeply into the region below the zero streamline as M increases Although the quadratic profile does not give a highly accurate description of the details of the profiles in the region of small velocity, it demonstrates qualitatively the large increase in total shear layer thickness with increasing Mach number

5) The use of a "proper" length scale for the characteristic thickness of the initial shear is important In Figs 1 and 2, the momentum thickness $\theta_0 \equiv \bar{\theta}_0$ is used to define the streamwise coordinate $\xi = (\nu s/u \theta_0^2)$ Figure 2 shows that the Mach number ratio (M_*/M) achieves one half the value for $\xi \to \infty$ at about $\xi \cong 0\,2$, and this result is independent of M_e and $H(-\delta_2)$ An equivalent result is found for the growth of the shear layer velocity thickness $\bar{\delta} = \bar{\delta}_1 + \bar{\delta}_2$ However, if $\bar{\delta}$ is plotted as a function of $(\nu s/u \bar{\delta}_0^2)$ as in Fig 5, the value of the streamwise coordinate for which $(\bar{\delta}/\bar{\delta}_0)$ approaches the similarity law $(\bar{\delta}/\bar{\delta}_0) \sim (\nu s/u \bar{\delta}_0^2)^{1/2}$ varies both with M and $H(-\delta_2)$

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¶ The pressure p is the static pressure along the outer edge of the shear layer, and p' is the static pressure downstream of the neck of the wake

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A Study of Wakes behind a Circular Cylinder at $M = 5.7$

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The flow field behind a circular cylinder was investigated experimentally at a nominal Mach number of 5 7, over a range of Reynolds numbers from 4500 to 66,500, based on the cylinder diameter Pitot pressure, static pressure, and total temperature were measured at various distances behind three cylinders of different diameters in order to determine the flow properties in the wake The data at different Reynolds numbers were correlated in a manner indicated by a linearized theory for the laminar far wake with axial pressure gradient Transition from laminar to turbulent flow was determined from the velocity profiles and correlated with results obtained from mass-diffusion and hot-wire fluctuation measurements

Nomenclature

d = cylinder diameter
h = static enthalpy
M = Mach number
p = static pressure
p_0 = freestream stagnation pressure, absolute
p_{0g} = freestream stagnation pressure, gage
p_p = pitot pressure
Re_d = freestream Reynolds number based on cylinder diameter
Re_x = Reynolds number based on distance along wake and conditions at edge of boundary layer
Re_T = Townsend Reynolds number
T_{aw} = adiabatic wire temperature
u = component of velocity parallel to wake centerline
v = component of velocity normal to wake centerline
w = velocity defect, $1 - (u/u_e)$
x = length along wake centerline measured from center of cylinder
y = length normal to wake centerline measured from line through center of cylinder
ϵ_T = turbulent diffusivity
ω = angle of cylinder measured from stagnation point

Subscripts

e = conditions at outer edge of wake
m = measured
0 = initial conditions (usually taken at wake neck)

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Introduction

THE nature of wakes is one of the oldest basic problems in the field of classical fluid mechanics Although the wakes in low-speed flow have been discussed in several treatises,[1–6] it was only recently that research was directed toward the phenomenon of high-speed flow The development of intercontinental ballistic missiles and hypersonic entry vehicles has kindled this interest in high-speed wakes

The theoretical problem of hypersonic wakes was first treated by Feldman,[7] who devised a simple model of the wake flow Feldman's basic approach has been extended by Lykoudis[8] More recently, Lees and Hromas[9] have attacked the problem of turbulent diffusion in the wake by using integral methods to solve the boundary-layer equations The two-dimensional laminar hypersonic wake with streamwise pressure gradient has been solved by Kubota,[10] using linearized equations

Perhaps the major obstacle in correlating theory and experiment in hypersonic wakes is the difficulty of obtaining reliable and detailed experimental data Atmospheric entry involves a range of temperature and Mach number that cannot be simulated effectively by known devices for extended periods Therefore, only limited data have been secured, either from short-duration experimental facilities, such as ballistic ranges[11] and shock tunnels[12] or from actual flight where results are not easily duplicated The conventional wind tunnel does not simulate the high Mach